

On boundedness of solutions of reaction–diffusion equations with nonlinear boundary conditions*

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Abstract

We give conditions on the nonlinearities of a reaction diffusion equation with nonlinear boundary conditions that guarantee that any solution is bounded locally around certain point x_0 of the boundary, uniformly for all positive time. The conditions imposed are of local nature and need only to hold in a small neighborhood of the point x_0 .

Keywords: Reaction-diffusion, nonlinear boundary conditions, bounded solutions, blow-up

1 Introduction

In this article we consider the following reaction diffusion equation with nonlinear boundary conditions in a smooth domain $\Omega \subset \mathbb{R}^N$,

$$\begin{cases} u_t - \Delta u = f(x, u) & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_D \\ \frac{\partial u}{\partial \vec{n}} = g(x, u) & \text{on } \Gamma_N \\ u(0, x) = u_0(x) \geq 0 & \text{in } \Omega \end{cases} \quad (1.1)$$

where $\Gamma = \partial\Omega = \Gamma_D \cup \Gamma_N$ is a disjoint partition of the boundary of Ω and f and g are suitably smooth functions of (x, u) . The subindices D and N on Γ indicate the part of the boundary with Dirichlet and Neumann type condition, respectively. We are interested in nonnegative solutions of (1.1) so we will assume

$$f(x, 0) \geq 0, \text{ for all } x \in \Omega, \quad g(x, 0) \geq 0 \text{ for all } x \in \Gamma_N$$

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We want to obtain local conditions on the nonlinearities f and g , which will be imposed in a neighborhood of a point $x_0 \in \partial\Omega$, that guarantee that for any initial condition $u_0 \in L^\infty(\Omega)$, the proper solution of problem (1.1) starting at u_0 (which is defined for all $t > 0$), is bounded in the neighborhood of x_0 uniformly for all $t > 0$. See Section 2 for an appropriate definition of the concept of “proper solution”.

As a matter of fact, we will be able to prove the following result

Theorem 1.1 *Let $x_0 \in \Gamma_N$, $p > 1$, $q \geq 1$ and let $R_0 > 0$, $M_0 > 0$ such that*

$$\begin{aligned} f(x, u) &\leq -\beta_0 u^p, & x \in B(x_0, R_0) \cap \Omega, & u \geq M_0 \\ g(x, u) &\leq u^q & x \in B(x_0, R_0) \cap \partial\Omega, & u \geq M_0 \end{aligned} \quad (1.2)$$

If one of the two following conditions holds

i) $p + 1 > 2q$ and $\beta_0 > 0$ or

ii) $p + 1 = 2q$ and $\beta_0 > q$,

then, for any initial condition $0 \leq u_0 \in L^\infty(\Omega)$ the proper minimal solution of (1.1) starting at u_0 is bounded in a neighborhood of x_0 in $\bar{\Omega}$, for all $t > 0$. That is, there exist $\delta, M > 0$ such that

$$\sup_{0 \leq t < \infty, x \in B(x_0, \delta) \cap \bar{\Omega}} u(t, x, u_0) \leq M \quad (1.3)$$

We want to stress the local nature of the result: the conditions imposed on the nonlinearities are localized around a neighborhood of $x_0 \in \partial\Omega$ and the result concludes the boundedness of the solution around the point x_0 . This boundedness around x_0 is obtained regardless of the behavior of the nonlinearities and/or the solution away from this point.

This result is, in some sense, the complementary result of [4], in which the authors proved that if the conditions are reversed then blow-up occurs at this point of the boundary. Among other things, they were able to show (see [4]) that if the nonlinearities satisfy

$$f(x, u) \geq -\beta_0 u^p, \quad g(x, u) \geq u^q$$

locally around $x_0 \in \partial\Omega$ and for u large enough and if

i) $p + 1 < 2q$ or

ii) $p + 1 = 2q$ and $\beta_0 < q$,

then there exists initial conditions with support near x_0 such that the solution starting at this initial condition blows up in finite time in a neighborhood of $x_0 \in \partial\Omega$. Again, this blow-up is produced regardless of the behavior of the nonlinearities or the solution away from this point.

Remark 1.2 *i) With an appropriate rescaling it is not difficult to see that if the local conditions of the nonlinearities f and g in Theorem 1.1 are of the type $f(x, u) \leq -\beta_0 u^p$, $g(x, u) \leq \alpha_0 u^q$, for $x \in B(x_0, R_0) \cap \partial\Omega$, $u \geq M_0$, then, the condition $\beta_0 > q$ in ii) should be changed to $\beta_0 > q\alpha_0^2$.*

ii) Similarly, if in [4], the nonlinearity at the boundary satisfy $g(x, u) \geq \alpha_0 u^q$ then blow-up is obtained under the condition $p + 1 < 2q$, $\alpha_0 > 0$ or $p + 1 = 2q$ and $\beta_0 < q\alpha_0^2$.

iii) It is interesting to note that the relations obtained are independent of the domain or the dimension.

As an example, consider for instance the problem

$$\begin{cases} u_t - \Delta u = -\beta(x)u^p & \text{in } \Omega \\ \frac{\partial u}{\partial n} = \alpha(x)u^q & \text{on } \partial\Omega \\ u(0, x) = u_0(x) \geq 0 & \text{in } \Omega \end{cases} \quad (1.4)$$

with β and α continuous functions, $\beta(x) > 0$ in $\bar{\Omega}$ and $\alpha(x) > 0$ in $\partial\Omega$. Then if $p + 1 = 2q > 2$ and $x_0 \in \partial\Omega$ with $\frac{\beta(x_0)}{\alpha(x_0)^2} < q$ then from [4], there are initial conditions where blow up is produced near x_0 , while if $\frac{\beta(x_0)}{\alpha(x_0)^2} > q$, then from Theorem 1.1 above, for any initial condition $u_0 \in L^\infty(\bar{\Omega})$ the proper minimal solution is bounded near x_0 . Hence, we have the situation as in Figure 1.

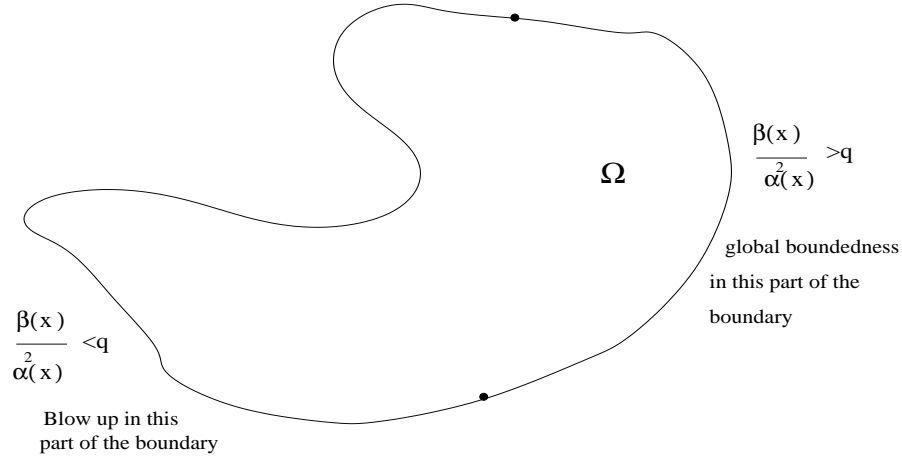


Figure 1.

Nowadays, there exists an extensive literature dealing with the dynamic behavior of the solutions of problem (1.1) and the characterization of blow-up or boundedness of solutions according to the behavior of the nonlinearities for large u . In the pioneer work of [7] they treated the one dimensional case, say $\Omega = (0, 1)$, with $f(x, u) = -\beta u^p$, $g(x, u) = u^q$ and $\Gamma_D = \emptyset$ and they already obtained that the critical relations are $p + 1$ vs. $2q$ and if $p + 1 = 2q$ then β vs. q , in the sense that if $p + 1 < 2q$ or $p + 1 = 2q$ and $\beta < q$ then blow-up is produced and if $p + 1 > 2q$ or $p + 1 = 2q$ and $\beta > q$ then the solutions are globally bounded. They also treated the very delicate case where $p + 1 = 2q$ and $\beta = q$. They actually showed that the solutions were defined for all time $t > 0$ but the phenomena of infinite time blow-up was present.

Later on, in [13, 14], they treated the case of arbitrary dimension and obtained that if $\Gamma_D = \emptyset$ and the nonlinearities f and g that behave for u large as $f \sim -\beta u^p$ and $g \sim u^q$, then blow-up is produced if $p + 1 < 2q$ or if $p + 1 = 2q$ and $\beta < q$. Also, they showed

that if $p + 1 > 2q$ or if $p + 1 = 2q$ and β is large enough, then the solutions are globally bounded. Also, in [1] they studied the porous medium equation in any dimension and as a particular case they considered the equation (1.1) with $\Gamma_D = \emptyset$, $f(x, u) = -\beta u^p$ and $g(x, u) = u^q$. They showed that if $p + 1 < 2q$ or $p + 1 = 2q$ and $\beta < q$ then blow-up is produced and if $p + 1 > 2q$ or $p + 1 = 2q$ and $\beta > q$ then the solutions are globally bounded.

With all these works it is clear that the critical relations that mark the line between blow-up and boundedness are given by $p + 1$ vs. $2q$ and in case $p + 1 = 2q$, β vs. q . These works have a common characteristic and it is that the nonlinear boundary condition is imposed in the whole domain, $\Gamma_D = \emptyset$ and the construction of sub or super solutions is done for the whole domain. Hence, the relations between p , q and β need to hold throughout the domain to obtain the result and either the blow-up result or the boundedness result obtained are global in space. In particular, none of them can treat the case as in the equation (1.4) where $p + 1 = 2q$ but in some part of the boundary the relation is $\beta > q$ and in other part the relation is $\beta < q$.

In this direction, the work [4], shows that the relations between p , q and β that produce blow-up are local. That is, if the relations $p + 1 < 2q$ or $p + 1 = 2q$ and $\beta < q$ hold in a neighborhood of a point $x_0 \in \partial\Omega$ then, blow up is produced at that point of the boundary and actually the blow up is obtained in $B(x_0, \delta) \cap \partial\Omega$ for some δ small enough.

In this work we show that if the relations between p , q and β that guarantee boundedness of solutions hold just in a small neighborhood of a point of the boundary, then, the solution remains bounded in a small neighborhood of x_0 .

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2 A comment on proper solutions

Observe that Theorem 1.1 concludes that the solution of problem (1.1) is bounded near a point $x_0 \in \partial\Omega$ uniformly for all $t > 0$. To be able to prove this result we will need first a definition of the solution of (1.1) which is defined for all $t > 0$. An appropriate way to accomplish this is with the concept of “proper solution”, which goes back to [6] and it has been further developed by [8, 9], mainly for homogeneous Dirichlet boundary conditions. See also [12]. This concept of “proper solution” has been proved to be an appropriate way to extend a solution beyond the time of explosion (in case it blows up) and therefore being able to say something about the solution beyond this time.

The idea behind this concept is to approximate the nonlinearities f and g monotonically from below with a sequence f_n , g_n such that the solution, $u_n(x, t, u_0)$, of the approximant problem

$$\begin{cases} u_t - \Delta u = f_n(x, u) & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_D \\ \frac{\partial u}{\partial \vec{n}} = g_n(x, u) & \text{on } \Gamma_N \\ u(0, x) = u_0(x) \geq 0 & \text{in } \Omega \end{cases} \quad (2.1)$$

are globally defined in time. Using monotonicity arguments, we have that for fixed x, t the sequence $\{u_n(x, t, u_0)\}_n$ is monotone increasing. Hence it will converge to $u(x, t, u_0) \in [0, +\infty]$ (notice that we do not exclude the fact that u maybe $+\infty$). This function, by definition, is the proper solution.

For equation (1.1) we define these solutions as follows: let $f_n(x, u) = \min\{f(x, u), n\}$ and $g_n(x, u) = \min\{g(x, u), n\}$. Notice that if f and g are locally Lipschitz functions in u uniformly in x , that is, for each $R > 0$ we have the existence of $L(R)$ such that

$$|f(x, u) - f(x, v)| \leq L(R)|u - v|, \quad |u|, |v| \leq R, \quad x \in \bar{\Omega} \quad (2.2)$$

and similarly for g , then f_n and g_n satisfy also the same Lipschitz relation (2.2). This condition and the fact that the functions f_n and g_n are bounded above by n imply (see [2, 3]) that the solutions of (2.1) are defined for all $t > 0$.

Now, it is not difficult to see by monotonicity arguments that, since $f_n \leq f_{n+1}$ and $g_n \leq g_{n+1}$, we have $0 \leq u_n(x, t, u_0) \leq u_{n+1}(x, t, u_0)$ and therefore $u_n(x, t, u_0) \nearrow u(x, t, u_0)$ as $n \rightarrow \infty$. Moreover if $u(x, t, u_0) < \infty$ for $(x, t) \in [0, T] \times \bar{\Omega}$ then it coincides with the classical solution for $t \in [0, T]$. Also, using monotonicity arguments, it is not difficult to see that the function u is independent of the monotone sequences f_n and g_n chosen to approximate f and g .

In this paper, whenever we refer to the proper solution of problem (1.1) we refer to the solution constructed in this way.

3 Proof of the main result

In this section we will provide a proof of our main result, Theorem 1.1. To accomplish this, we will construct appropriate super solutions locally around the point $x_0 \in \Gamma_N$. As a matter of fact we will extensively use the singular solutions of the following elliptic problem

$$\begin{cases} -\Delta z + \beta z^p = 0, & B(0, R) \\ z(R) = +\infty \end{cases} \quad (3.1)$$

and the fact that the asymptotics of this radial solution as $r \rightarrow R$ is very well understood, see [5, 11].

This solution and more exactly, an appropriate approximation of it, will be used as a supersolution near the boundary.

Before embarking in the proof of the main result, we will need some preliminary lemmas.

Lemma 3.1 *Let R, β and β_0 be positive numbers with $\beta < \beta_0$ and consider the function*

$$H_{R,\beta}(r) = C^* \frac{1}{\beta^{1/(p-1)}} (R - r)^{-\frac{2}{p-1}} \quad (3.2)$$

defined for $0 < r < R$, where the constant

$$C^* = \left(\frac{2(p+1)}{(p-1)^2} \right)^{\frac{1}{p-1}} \quad (3.3)$$

Then, if $\delta \in (0, 1)$ is given by

$$\frac{1}{1-\delta} = 1 + \frac{\beta_0 - \beta}{\beta} \cdot \frac{(C^*)^{p-1}}{N-1}$$

we have

$$-\Delta H_{R,\beta} + \beta_0(H_{R,\beta})^p \geq 0, \text{ for } (1-\delta)R < r < R \quad (3.4)$$

Proof. The proof is just a direct computation. If, to simplify, we denote by H the function $H_{R,\beta}$, observe that $-\Delta H + \beta_0 H^p = -H'' - \frac{N-1}{r}H' + \beta_0 H^p$, where we use the notation $' = d/dr$.

Direct calculations show that

$$\begin{aligned} H'' &= \beta H^p, \\ \left| \frac{N-1}{r} H' \right| &\leq \frac{N-1}{r} \cdot (C^*)^{1-p} \beta (R-r) H^p. \end{aligned}$$

In particular,

$$-\Delta H + \beta_0 H^p \geq (\beta_0 - \beta(1 + \frac{N-1}{r}(C^*)^{1-p}(R-r)))H^p.$$

Hence, $\beta_0 - \beta(1 + \frac{N-1}{r}(C^*)^{1-p}(R-r)) \geq 0$ if

$$\frac{R-r}{r} \leq \frac{\beta_0 - \beta}{\beta} \cdot \frac{(C^*)^{p-1}}{N-1},$$

which is equivalent to say that $(1-\delta)R < r < R$ with

$$\frac{1}{1-\delta} = 1 + \frac{\beta_0 - \beta}{\beta} \cdot \frac{(C^*)^{p-1}}{N-1}$$

Remark 3.2 Notice that $H_{R,\beta}(r) = \frac{1}{(R^2\beta)^{\frac{1}{p-1}}} H_{1,1}(\frac{r}{R})$, $0 < r < R$.

Lemma 3.3 For $R, \beta > 0$, define the function $z_{R,\beta}$ as the unique radial solution of the singular problem

$$\begin{cases} -\Delta z + \beta z^p = 0, & B(0, R), \\ z(R) = +\infty. \end{cases} \quad (3.5)$$

Then, we have

$$z_{R,\beta}(r) = \frac{1}{(R^2\beta)^{\frac{1}{p-1}}} z_{1,1}(\frac{r}{R}), \quad 0 < r < R. \quad (3.6)$$

Moreover for each fixed $\gamma \in (0, 1)$ small there exists $\eta > 0$, also small, such that

$$z_{1,1}(r) \leq H_{1,1-\gamma}(r), \quad 1 - \eta < r < 1. \quad (3.7)$$

In particular, we have

$$z_{R,\beta}(r) \leq H_{R,(1-\gamma)\beta}(r), \quad (1 - \eta)R < r < R. \quad (3.8)$$

Proof. Statement (3.6) follows from a standard scaling argument.

To prove (3.7) notice that from [5, 11] the function $z_{1,1}$ has the following expression

$$z_{1,1}(r) = C^*(1 - r)^{-2/(p-1)}(1 + O(1 - r))$$

Hence, if we fix $\gamma \in (0, 1)$, we have that

$$\frac{z_{1,1}(r)}{H_{1,1-\gamma}(r)} = \frac{C^*(1 - r)^{-2/(p-1)}(1 + O(1 - r))}{C^*(1 - \gamma)^{\frac{1}{1-p}}(1 - r)^{-2/(p-1)}} = \frac{1 + O(1 - r)}{(1 - \gamma)^{\frac{1}{1-p}}},$$

which implies that $z_{1,1}(r) \leq H_{1,1-\gamma}(r)$, for r near 1. This shows (3.7). To show (3.8) we just use (3.6) and the fact that we also have $H_{R,\beta}(r) = \frac{1}{(R^2\beta)^{\frac{1}{p-1}}}H_{1,1}(\frac{r}{R})$ for $0 < r < R$, see Remark 3.2. \square

We are in a position now to prove the main result

Proof of Theorem 1.1: Let $x_0 \in \partial\Omega$ and assume the hypotheses of the theorem. In order to write a proof that will work for both cases, i) and ii), let us denote by $\hat{q} = 0$ if we are in case i) or $\hat{q} = q$ if we are in case ii), so that we have that $\beta_0 > \hat{q}$.

Consider now $R^* < R_0/2$, small enough such that, for all $R \leq R^*$, if $y_R = x_0 - R\vec{n}(x_0)$, where $\vec{n}(x)$ is the unit exterior normal vector at the point $x \in \partial\Omega$, then $B(y_R, R) \subset \Omega$ and $\partial\Omega \cap \bar{B}(y_R, R) = \{x_0\}$, see Figure 2.

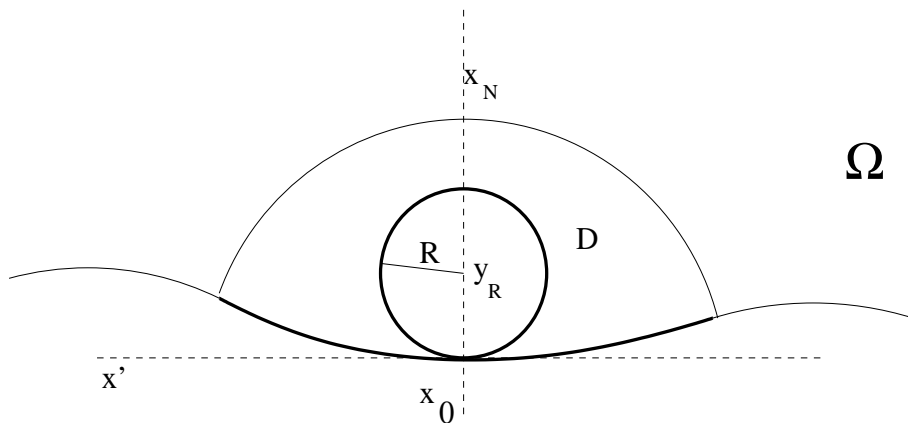


Figure 2.

Consider the following

Lemma 3.4 *If $u_0 \in L^\infty(\Omega)$ and if $u(t, x, u_0)$ is the proper solution of (1.1) then there exists $0 < R_1 \leq R^*$ such that for all $R < R_1$ we have*

$$0 \leq u(t, x, u_0) \leq z_{R, \beta_0}(|x - y_R|), \text{ for } (t, x) \in [0, \infty) \times B(y_R, R) \quad (3.9)$$

Proof. It is very easy to see that $z_{R, \beta_0}(r) \geq z_{R, \beta_0}(0) = \frac{1}{(R^2 \beta_0)^{\frac{1}{p-1}}} z_{1,1}(0) \rightarrow +\infty$ as $R \rightarrow 0$. Let us choose, $0 < R_1 \leq R^*$, small enough, such that for all $0 < R \leq R_1$

$$u_0(x) + 1 \leq z_{R, \beta_0}(|x - y_R|), \text{ for } |x - y_R| \leq R.$$

Hence, since the function $z_{R, \beta}$ is obtained as the limit as $n \rightarrow +\infty$ of the functions z_{R, β_0}^n which is the unique radial solution of

$$\begin{cases} -\Delta z + \beta_0 z^p = 0, & B(0, R) \\ z(R) = n, \end{cases} \quad (3.10)$$

then we can choose n_0 such that for all $n \geq n_0$ we have

$$u_0(x) + 1/2 \leq z_{R, \beta_0}^n(|x - y_R|), \text{ for } |x - y_R| \leq R, \quad n \geq n_0.$$

Let us fix $T > 0$ and let $u_m(t, x, u_0)$ be an aproximants of the proper solution $u(t, x, u_0)$. From comparison principles, it is easy to see that for fixed $m > 0$, we have $n = n(m)$, large enough, such that

$$u_m(t, x, u_0) \leq z_{R, \beta_0}^n(|x - y_R|), \quad 0 \leq t \leq T, \quad x \in B(y_R, R), \quad n \geq n(m).$$

Passing to the limit as $n \rightarrow +\infty$ first and $m \rightarrow \infty$ second we get

$$u(t, x, u_0) \leq z_{R, \beta_0}(|x - y_R|), \quad 0 \leq t \leq T, \quad x \in B(y_R, R)$$

Since $T > 0$ is arbitrary we have the above inequality for all $t > 0$. This implies the result. \square

Now, if we consider β_1 , with $\hat{q} < \beta_1 < \beta_0$ and we consider $\gamma = 1 - \frac{\beta_1}{\beta_0}$ we have from (3.8) and (3.9) that

$$u(t, x, u_0) \leq z_{R, \beta_0}(|x - y_R|) \leq H_{R, \beta_1}(|x - y_R|), \text{ for } r_0 < |x - y_R| < R$$

where $r_0 = (1 - \eta)R$ as it is proved above.

Moreover if we consider now β_2 , with $\hat{q} < \beta_2 < \beta_1 < \beta_0$, we have

$$H_{R, \beta_1}(|x - y_R|) < H_{R, \beta_2}(|x - y_R|), \quad 0 < |x - y_R| < R$$

Hence, if we apply Lemma 3.1 with $\beta = \beta_2$, we have that if ρ is such that $r_0 < \rho < R$, $(1 - \delta)(R + \varepsilon_0) < \rho < R$ where δ is given by Lemma 3.1 and $\varepsilon_0 > 0$ is small enough, then for all $0 < \varepsilon \leq \varepsilon_0$ we obtain

$$u(t, x, u_0) < H_{R+\varepsilon, \beta_2}(\rho), \quad |x - y_R| = \rho$$

Therefore, for all $0 < \varepsilon < \varepsilon_0$, if we define $v_\varepsilon(x) = H_{R+\varepsilon, \beta_2}(|x - y_R|)$ for $|x - y_R| < R + \varepsilon$, we have the following, (see Figure 3).

$$\begin{cases} -\Delta v_\varepsilon + \beta_0 v_\varepsilon^p \geq 0, & \Omega \cap \{\rho < |x - y_R| < R + \varepsilon\} \\ v_\varepsilon(x) \geq u(t, x, u_0), & \text{for } |x - y_R| = \rho \\ v_\varepsilon(x) = +\infty \geq u(t, x, u_0), & \Omega \cap \{|x - y_R| = R + \varepsilon\} \end{cases} \quad (3.11)$$

Hence, we just need to show that in $\partial\Omega \cap \{|x - y_R| \leq R + \varepsilon\}$ we have $\frac{\partial v_\varepsilon}{\partial n} \geq v_\varepsilon^q$. To show this, observe that if $x \in \partial\Omega \cap \{|x - y_R| \leq R + \varepsilon\}$, if $r = |x - y_R|$, then

$$\frac{\partial v_\varepsilon}{\partial n} = \nabla v_\varepsilon(x) \cdot \vec{n}(x) = H'_{R+\varepsilon, \beta_2}(r) \frac{x - y_R}{|x - y_R|} \cdot \vec{n}(x)$$

But, from the smoothness of the boundary we have that the scalar product $\frac{x - y_R}{|x - y_R|} \cdot \vec{n}(x)$ is very close to 1 if ε is small. Moreover, direct computations show that

$$H'_{R+\varepsilon, \beta_2}(r) = \sqrt{\frac{2\beta_2}{p+1}} (H_{R+\varepsilon, \beta_2}(r))^{\frac{p+1}{2}}$$

Hence,

$$\frac{\partial v_\varepsilon}{\partial n} \geq \sqrt{\frac{2\beta_2}{p+1}} \frac{x - y_R}{|x - y_R|} \cdot \vec{n}(x) v_\varepsilon^{\frac{p+1}{2}}, \quad x \in \partial\Omega \cap B(y_R, R + \varepsilon) \quad (3.12)$$

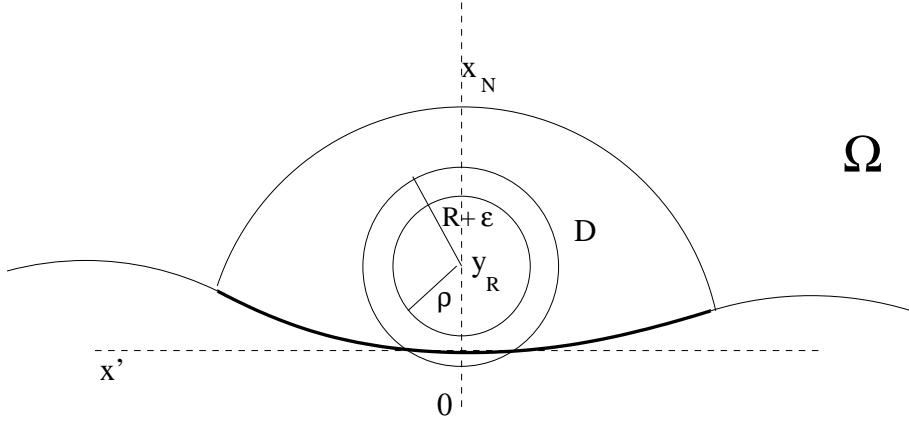


Figure 3.

We distinguish now the two possible cases.

If we are in case i) then $\frac{p+1}{2} > q$, $\beta_2 > 0$ and we can write inequality (3.12) as

$$\frac{\partial v_\varepsilon}{\partial n} \geq \left[\sqrt{\frac{2\beta_2}{p+1}} \frac{x - y_R}{|x - y_R|} \cdot \vec{n}(x) v_\varepsilon^{\frac{p+1}{2}-q} \right] v_\varepsilon^q, \quad x \in \partial\Omega \cap B(y_R, R + \varepsilon)$$

But if $x \in \partial\Omega \cap B(y_R, R + \varepsilon)$ we have $R \leq |x - y_R| \leq R + \varepsilon$ and since the function H is increasing with the distance to y_R we have

$$v_\varepsilon(x) \geq H_{R+\varepsilon, \beta_2}(R) = C^* \frac{1}{\beta_2^{\frac{1}{p-1}}} \varepsilon^{\frac{-2}{p-1}} \rightarrow +\infty \text{ as } \varepsilon \rightarrow 0$$

Hence, since for ε small enough we have $\frac{x-y_R}{|x-y_R|} \cdot \vec{n}(x) \geq 1 - \delta(\varepsilon)$ and $\delta(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$ we get

$$\frac{\partial v_\varepsilon}{\partial n} \geq \left[\sqrt{\frac{2\beta_2}{p+1}} (1 - \delta(\varepsilon)) \left[C^* \frac{1}{\beta_2^{\frac{1}{p-1}}} \varepsilon^{\frac{-2}{p-1}} \right]^{\frac{p+1}{2}-q} \right] v_\varepsilon^q, \quad x \in \partial\Omega \cap B(y_R, R + \varepsilon)$$

Choosing ε small enough we can guarantee that

$$\sqrt{\frac{2\beta_2}{p+1}} (1 - \delta(\varepsilon)) \left[C^* \frac{1}{\beta_2^{\frac{1}{p-1}}} \varepsilon^{\frac{-2}{p-1}} \right]^{\frac{p+1}{2}-q} \geq 1$$

and this implies that $\frac{\partial v_\varepsilon}{\partial n} \geq v_\varepsilon^q$ for $x \in \partial\Omega \cap B(y_R, R + \varepsilon)$. This shows the result for the first case.

If we are in case ii) then $\frac{p+1}{2} = q$, $\beta_2 > q$ and we can write (3.12) as

$$\frac{\partial v_\varepsilon}{\partial n} \geq \left[\sqrt{\frac{\beta_2}{q}} \frac{x - y_R}{|x - y_R|} \vec{n}(x) \right] v_\varepsilon^q, \quad x \in \partial\Omega \cap B(y_R, R + \varepsilon)$$

But as in case i) we have $\frac{x-y_R}{|x-y_R|} \vec{n}(x) \geq 1 - \delta(\varepsilon)$ and $\delta(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$. Hence, since $\beta_2 > q$, for ε small enough we have

$$\frac{\partial v_\varepsilon}{\partial n} \geq \left[\sqrt{\frac{\beta_2}{q}} (1 - \delta(\varepsilon)) \right] v_\varepsilon^q \geq v_\varepsilon^q, \quad x \in \partial\Omega \cap B(y_R, R + \varepsilon)$$

This completes the proof of the theorem. \square

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